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# Baxter's $Q$-operator for the homogeneous $XXX$ spin chain

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**Abstract.** Applying the Pasquier–Gaudin procedure we construct Baxter's $Q$ operator for the homogeneous $XXX$ model as an integral operator in the standard representation of $SL(2)$. The connection between the $Q$ operator and the local Hamiltonians is discussed. We show that Lipatov's duality symmetry operator arises naturally as the leading term of the asymptotic expansion of the $Q$ operator for large values of the spectral parameter.

## 1. Introduction

The modern approach to the theory of integrable systems is given by the quantum inverse scattering method (QISM) [4, 7]. In the framework of QISM, the eigenstates $|\lambda_1, \ldots, \lambda_l\rangle$ are obtained by the algebraic Bethe ansatz (ABA) method as excitations over the vacuum state and the spectral problem is reduced to the set of algebraic Bethe equations (BE) for the parameters $\lambda_j$. In fact the ABA is equivalent to the construction of the eigenfunctions in a special representation as polynomials of some suitable variables.

The alternative approach is the $Q$-operator method [1] proposed by Baxter: an operator $\hat{Q}(\lambda)$ exists which obeys Baxter's equation. The set of Bethe equations is equivalent to Baxter's equation for the eigenvalue $Q(\lambda)$ of the $Q$ operator. This second-order, finite-difference equation is the simple consequence of Baxter's relation for the transfer matrix and the $Q$ operator [1].

The ABA and $Q$-operator method are equivalent when eigenfunctions and therefore $Q(\lambda)$ are polynomials. In the more general 'non-polynomial' situation one could use the $Q$-operator method. The $Q$ operator for the periodic Toda chain was constructed by Pasquier and Gaudin [2]. The application of the $Q$ operator for the construction of eigenstates with arbitrary complex values of conformal weights in the $XXX$ spin-chain case was considered by Korchemsky and Faddeev [7]. In the present paper we construct the $Q$ operator for the homogeneous $XXX$ spin chain using the Pasquier–Gaudin procedure.

The paper is organized as follows. In section 2 we introduces definitions and the standard facts about Baxter's equation and the construction of local Hamiltonians. In section 3 we construct the $Q$ operator and study some properties of the $Q$ operator obtained in the simplest case of a homogeneous chain. In section 4 we obtain the connection between the $Q$ operator and the local Hamiltonian. In section 5 we consider the asymptotic expansion of the $Q$ operator for large spectral parameter. The duality symmetry operator introduced by Lipatov [9] appears naturally as the leading term in this asymptotic. Finally, in section 6 we summarize our results.

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## 2. The $XXX$ spin chain

In this section we collect some basic facts about the $XXX$ spin chain.

### 2.1. The $R$ matrix and the Yang–Baxter equation

The main object is the so-called $R$ matrix which is the solution of the Yang–Baxter equation

$$R_{12}(\lambda)R_{13}(\lambda+\mu)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda+\mu)R_{12}(\lambda). \tag{2.1.1}$$

The operator $R_{ij}(\lambda)$ depends on some complex variable: the spectral parameter $\lambda$ and two sets of $SL(2)$-generators $\vec{S}_i$ and $\vec{S}_j$ acting in different vector spaces $V_i$ and $V_j$.

Fixing the representations of the spins $s_i$ and $s_j$ in the vector spaces $V_i$ and $V_j$, we obtain the following $R$ matrices.

- $s = 1/2$ in space $V_i$ and arbitrary representation $s$ for the $V_j$:

  $$R_j(\lambda) = \lambda + \tfrac{1}{2}\eta + \eta \cdot \vec{S}_j\vec{\sigma}.$$

  This $R$ matrix is used for the construction of the Lax $L$ operator:

  $$L_i(\lambda) \equiv R\left(\lambda - \frac{\eta}{2}\right) = \lambda + \eta \cdot \vec{S}_i\vec{\sigma} = \begin{pmatrix} \lambda + \eta S_i & \eta S_i^- \\ \eta S_i^+ & \lambda - \eta S_i \end{pmatrix}. \tag{2.1.2}$$

- The equivalent representations $s$ in the spaces $V_i$ and $V_j$ [3]:

  $$R_{ij}(\lambda) = P_{ij} \cdot \frac{\Gamma(J_{ij} + \eta\lambda)}{\Gamma(J_{ij} - \eta\lambda)} \qquad J_{ij} \cdot (J_{ij} - 1) = L_{ij} \tag{2.1.3}$$

  where $P_{ij}$ is the permutation and $L_{ij}$ is the 'two-particle' Casimir in $V_i \otimes V_j$. This fundamental $R$ matrix is the building block for the construction of the local Hamiltonians.

### 2.2. Baxter's equation for the $XXX$ model.

The 'usual' quantum monodromy matrix $T(\lambda)$ is defined as the product of the $L$ matrices in the common two-dimensional auxiliary space. $T(\lambda)$ is a $2 \times 2$ matrix with operator entries acting in the quantum space $\otimes_{i=1}^n V_i$:

$$T(\lambda) \equiv L_1(\lambda + c_1)L_2(\lambda + c_2)\cdots L_n(\lambda + c_n) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{2.2.1}$$

The quantum transfer matrix $t(\lambda)$ is obtained by taking the trace of $T(\lambda)$ in the auxiliary space:

$$t(\lambda) \equiv \mathrm{Tr}\, T(\lambda) = A(\lambda) + D(\lambda). \tag{2.2.2}$$

Due to the Yang–Baxter equation the family of operators $t(\lambda)$ commutes, its $\lambda$ expansion begins with power $\lambda^n$ and provides $n - 1$ commuting operators $Q_k$:

$$t(\lambda) \cdot t(\mu) = t(\mu) \cdot t(\lambda) \qquad t(\lambda) = 2\lambda^n + \sum_{k=0}^{n-2} Q_k \lambda^k. \tag{2.2.3}$$

It is possible to show that the transfer matrix $t(\lambda)$ is $SL(2)$-invariant

$$[\vec{S}, t(\lambda)] = 0 \qquad \vec{S} \equiv \sum_{k=1}^{n} \vec{S}_k.$$

Therefore there exists the 'full' set of $n$ commuting operators: $n - 1$ operators $Q_k$ and the operator $S$. Due to $SL(2)$-invariance the subspace of the eigenvectors of the operator $t(\lambda)$ with eigenvalue $\tau(\lambda)$ is the $SL(2)$ module generated by the highest-weight vector $\Psi$, i.e. the vector space spanned by linear combinations of monomials in the $S^+$ applied to the vector $\Psi$. The highest-weight vector $\Psi$ is defined by the equation $S^-\Psi = 0$.

We shall work in the standard representation of the group $SL(2)$:

$$S\Psi(x) \equiv (cx+d)^{-2s}\Psi\left(\frac{ax+b}{cx+d}\right) \qquad S^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where the $SL(2)$-generators are given as differential operators:

$$S_k = x_k\partial_k + s_k \qquad S_k^- = -\partial_k \qquad S_k^+ = x_k^2\partial_k + 2s_k x_k \tag{2.2.4}$$

acting in the space of polynomials of the variable $x_k$. Here the 'spin' $s_k$ is an arbitrary number. In this representation the commuting operators $Q_k$ are 'local' differential operators acting in the space of polynomials of the $n$ variables $x_1, \dots, x_n$ and there exists the vacuum vector $|0\rangle$:

$$B(\lambda)|0\rangle = 0 \qquad A(\lambda)|0\rangle = \Delta_+(\lambda)|0\rangle \qquad D(\lambda)|0\rangle = \Delta_-(\lambda)|0\rangle$$

so that we can use the algebraic Bethe ansatz (ABA) method and reduce the problem of the common diagonalization of the operators $Q_k$ and $S$:

$$t(\lambda)\Psi_l = \tau(\lambda)\Psi_l \qquad S\Psi_l = \left(l + \sum_{k=1}^n s_k\right)\Psi_l$$

to the solution of the Bethe equation [4, 6]. The vacuum vector $|0\rangle$ is the common highest vector of the local representations of $SL(2)$:

$$|0\rangle \equiv \prod_{k=1}^n |0\rangle_k \qquad S_k^-|0\rangle_k = 0 \qquad S_k|0\rangle_k = s_k|0\rangle_k$$

and

$$L_k(\lambda + c_k)|0\rangle_k = \begin{pmatrix} \lambda + c_k + \eta s_k & 0 \\ \cdots & \lambda + c_k - \eta s_k \end{pmatrix}|0\rangle_k$$

so that

$$\Delta_\pm(\lambda) \equiv \prod_{k=1}^n (\lambda + c_k \pm \eta s_k). \tag{2.2.5}$$

Let us look now at the eigenvector $\Psi_l$ in the form

$$\Psi_l \equiv |\lambda_1, \dots, \lambda_l\rangle \equiv \prod_{j=1}^l \mathrm{C}(\lambda_j)|0\rangle \qquad S|\lambda_1, \dots, \lambda_l\rangle = \left(l + \sum_{k=1}^n s_k\right)|\lambda_1, \dots, \lambda_l\rangle.$$

It is possible to show that the vector $|\lambda_1, \dots, \lambda_l\rangle$ is an eigenvector of the operator $t(\lambda)$ with eigenvalue

$$\tau(\lambda) = \Delta_+(\lambda)\prod_{j=1}^l \frac{(\lambda - \lambda_j + \eta)}{(\lambda - \lambda_j)} + \Delta_-(\lambda)\prod_{j=1}^l \frac{(\lambda - \lambda_j - \eta)}{(\lambda - \lambda_j)} \tag{2.2.6}$$

on condition that the parameters $\lambda_i$ obey the Bethe equations

$$\prod_{j=1}^l (\lambda_i - \lambda_j + \eta)\Delta_+(\lambda_i) = \prod_{j=1}^l (\lambda_i - \lambda_j - \eta)\Delta_-(\lambda_i). \tag{2.2.7}$$

It also appears that Bethe vectors $|\lambda_1, \dots, \lambda_l\rangle$ are the highest-weight vectors

$$S^-|\lambda_1, \dots, \lambda_l\rangle = 0.$$

In the representation (2.2.4) the highest-weight vector $\Psi_l$ is represented by a homogeneous, translation-invariant polynomial of degree $l$ ($l = 0, 1, 2, \ldots$) in $n$ variables $x_1, \ldots, x_n$:

$$\sum_{k=1}^{n} x_k \partial_k \Psi_l(x_1, \ldots, x_n) = l\Psi_l(x_1, \ldots, x_n) \qquad \sum_{k=1}^{n} \partial_k \Psi_l(x_1, \ldots, x_n) = 0. \tag{2.2.8}$$

One can obtain the Bethe equation from the formula for $\tau(\lambda)$ by taking the residue at $\lambda = \lambda_i$ and using the fact that the polynomial $\tau(\lambda)$ is regular at this point. Finally, we see that equations (2.2.6), (2.2.7) are equivalent to Baxter's equation for the polynomial $Q(\lambda)$:

$$\tau(\lambda)Q(\lambda) = \Delta_+(\lambda)Q(\lambda+\eta) + \Delta_-(\lambda)Q(\lambda-\eta) \tag{2.2.9}$$

where

$$Q(\lambda) \equiv \text{constant} \times \prod_{j=1}^{l}(\lambda - \lambda_j). \tag{2.2.10}$$

### 2.3. Local Hamiltonians

Let us consider the homogeneous $XXX$ chain of equal spins: $c_k = 0$ and $s_k = s$ and fix the same representation $s$ in auxiliary space. In this case the quantum monodromy matrix $T_s(\lambda)$ is the product of the fundamental $R$ matrices (2.1.3):

$$T_s(\lambda) \equiv R_1(\lambda)R_2(\lambda)\cdots R_n(\lambda).$$

The transfer matrix $t_s(\lambda)$ is obtained by taking the trace of $T_s(\lambda)$ in the auxiliary space, and due to the Yang–Baxter equation the families of operators $t_s(\lambda)$ and $t(\lambda)$ commute:

$$t_s(\lambda) \equiv \mathrm{Tr}_s\, T_s(\lambda) \qquad t_s(\lambda)t_s(\mu) = t_s(\mu)t_s(\lambda) \qquad t(\lambda)t_s(\mu) = t_s(\mu)t(\lambda).$$

The $\lambda$ expansion of the $\log t_s(\lambda)$ provides the Hamiltonians $H_k$:

$$H_k \equiv \frac{1}{\eta}\frac{\partial^k}{\partial\lambda^k}\log t_s(\lambda)\bigg|_{\lambda=0} \qquad [H_k, H_l] = 0 \qquad [H_k, Q_l] = 0 \tag{2.3.1}$$

where the $k$th operator describes the interaction between $k+1$ nearest neighbours on the chain. Due to the evident equalities (see equations (2.1.3))

$$R_{ij}(0) = P_{ij} \qquad R'_{ij}(0) = 2\eta P_{ij}\cdot\psi(J_{ij})$$

one obtains the following expression for the first 'two-particle' Hamiltonian $H_1$:

$$H_1 = \sum_{k=1}^{n} \hat{H}_{k-1,k} \qquad \hat{H}_{k-1,k} = \frac{1}{\eta}P_{k-1,k}R'_{k-1,k} = 2\cdot\psi(J_{k-1,k})$$

where $\psi(x)$ is logarithmic derivative of $\Gamma(x)$. It is convenient to work with the 'shifted' Hamiltonian

$$H = \sum_{k=1}^{n} H_{k-1,k} \qquad H_{k-1,k} = 2\cdot\psi(J_{k-1,k}) - 2\psi(2s) \tag{2.3.2}$$

where the 'shift' constant is defined by the requirement

$$H_{k-1,k}|0\rangle = 0.$$

Let us calculate the eigenvalues of the operator $H_{k-1,k}$. The operator $H_{k-1,k}$ is $SL(2)$-invariant:

$$[S^{\pm}_{k-1} + S^{\pm}_k, H_{k-1,k}] = 0 \qquad [S_{k-1} + S_k, H_{k-1,k}] = 0$$

and its highest-weight eigenfunctions $\Psi_l$ have the following simple form in the representation (2.2.4):

$$(x_{k-1}\partial_{k-1} + x_k\partial_k)\Psi_l = l\Psi_l \qquad (\partial_{k-1} + \partial_k)\Psi_l = 0 \quad \Rightarrow \quad \Psi_l(x_{k-1}, x_k) = (x_{k-1} - x_k)^l.$$

The two-particle Casimir $L_{k-1,k}$ is the second-order differential operator

$$L_{k-1,k} = -(x_{k-1} - x_k)^{2-2s}\partial_{k-1}\partial_k(x_{k-1} - x_k)^{2s}$$

and its eigenvalues $L_l$ and the eigenvalues $J_l$ of the operator $J_{k-1,k}$ can easily be calculated:

$$L_l = (2s + l)(2s + l - 1) \qquad J_l = 2s + l.$$

Finally we obtain the eigenvalues $H_l$ of the operator $H_{k-1,k}$:

$$H_l = 2\psi(2s + l) - 2\psi(2s).$$

In the representation (2.2.4) the operator $H_{k-1,k}$ can be realized as some 'two-particle' integral operator acting on the variables $x_{k-1}$ and $x_k$:

$$\begin{aligned} \mathrm{H}_{k-1,k}\Psi(x_{k-1}, x_k) = -\int_0^1 \mathrm{d}\alpha\, \frac{\bar{\alpha}^{2s-1}}{\alpha}\big[\Psi(\bar{\alpha}x_{k-1} + \alpha x_k, x_k) + \Psi(x_{k-1}, \alpha x_{k-1} + \bar{\alpha}x_k) \\ - 2\Psi(x_{k-1}, x_k)\big] \end{aligned} \tag{2.3.3}$$

where $\bar{\alpha} \equiv 1 - \alpha$. Note that these integral operators arise naturally in QCD [8]. To prove the equality (2.3.3) it is sufficient to show that the eigenvalues of the integral operator coincide with the eigenvalues $H_l$:

$$-2\int_0^1 \mathrm{d}\alpha\, \frac{\bar{\alpha}^{2s-1}}{\alpha}\big[\bar{\alpha}^l - 1\big] = 2\big[\psi(2s + l) - \psi(2s)\big].$$

The expression for the eigenvalues of the full Hamiltonian $H$ can be found by the ABA method [5]:

$$H = \frac{1}{\eta}\sum_{j=1}^{l}\frac{\partial}{\partial\lambda_j}\log\frac{\lambda_j + \eta s}{\lambda_j - \eta s} = \frac{1}{\eta}\sum_{j=1}^{l}\left[\frac{1}{\eta s - \lambda_j} + \frac{1}{\eta s + \lambda_j}\right].$$

It is possible to rewrite this expression in terms of the $Q(\lambda)$ function (2.2.10) as follows:

$$H = \frac{Q'(\eta s)}{\eta Q(\eta s)} - \frac{Q'(-\eta s)}{\eta Q(-\eta s)}. \tag{2.3.4}$$

There exists an additional operator which commutes with the transfer matrix. It is the shift operator $P$:

$$P\Psi(z_1, z_2, \ldots, z_n) = \Psi(z_n, z_1, \ldots, z_{n-1}) \qquad P = t_s(0). \tag{2.3.5}$$

The eigenvalues of the shift operator $P$ can also be found by the ABA method [5]:

$$P_l = \prod_{j=1}^{l}\frac{\lambda_j - \eta s}{\lambda_j + \eta s} = \frac{Q(\eta s)}{Q(-\eta s)}. \tag{2.3.6}$$

In the following sections we shall construct Baxter's $Q$ operator and show that Baxter's equation (2.2.9) and equations (2.3.4), (2.3.6) arise from the corresponding relations for the $Q$ operator.

## 3. Baxter's *Q* operator

Baxter's $Q$ operator is the operator $\hat{Q}(\lambda)$ with the properties [1]:

- $t(\lambda)\hat{Q}(\lambda) = \Delta_+(\lambda)\hat{Q}(\lambda + \eta) + \Delta_-(\lambda)\hat{Q}(\lambda - \eta)$
- $\hat{Q}(\mu)\hat{Q}(\lambda) = \hat{Q}(\lambda)\hat{Q}(\mu)$
- $t(\mu)\hat{Q}(\lambda) = \hat{Q}(\lambda)t(\mu)$.

The operators $\hat{Q}(\lambda)$ and $t(\lambda)$ have the common set of eigenfunctions

$$\hat{Q}(\lambda)\Psi = Q(\lambda)\cdot\Psi \qquad t(\lambda)\Psi = \tau(\lambda)\cdot\Psi \tag{3.0.1}$$

and eigenvalues of these operators obey Baxter's equation (2.2.9). Note that the $Q$ function (2.2.10) can be naturally interpreted as the eigenvalue of the $Q$ operator.

We construct the operator $\hat{Q}(\lambda)$ in the standard representation of the group $SL(2)$ in the following form:

$$\hat{Q}(\lambda)\Psi(x) \equiv \langle RQ(\lambda; x, z)|\Psi(z)\rangle \qquad RQ(\lambda; z) \equiv z^{-2s}Q(\lambda; z^{-1}) \tag{3.0.2}$$

where $R$ is the transformation of inversion. The scalar product here is the standard $SL(2)$-invariant scalar product for functions of the one variable:

$$\langle\Psi(z)|\Phi(z)\rangle = \int_{|z|\leqslant 1} Dz\,\Psi(\bar{z})\Phi(z) \qquad Dz \equiv \frac{2s-1}{\pi}\frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{(1-\bar{z}z)^{2-2s}} \tag{3.0.3}$$

and $z$ is the 'integration' or 'dumb' variable ($\bar{z}$ is its complex conjugate). In (3.0.2) the scalar product over all variables $z_1, \ldots, z_n$ is assumed. The $SL(2)$-generators $S^{\pm}$ are conjugated with respect to this scalar product:

$$\langle\Psi|S^{\pm}\Phi\rangle = -\langle S^{\mp}\Psi|\Phi\rangle \qquad \langle\Psi|S\Phi\rangle = \langle S\Psi|\Phi\rangle.$$

Using the obvious identities

$$R\Phi(z) = z^{-2s}\Phi(z^{-1}) \qquad RS^{\pm}\Phi(z) = S^{\mp}R\Phi(z) \qquad RS\Phi(z) = -SR\Phi(z)$$

we obtain the following rules for transposition:

$$\begin{aligned}\langle RQ(\lambda; z)|S^{\pm}\Psi(z)\rangle &= -\langle RS^{\pm}Q(\lambda; z)|\Psi(z)\rangle\\ \langle Q(\lambda; z)|S\Psi(z)\rangle &= -\langle RSQ(\lambda; z)|\Psi(z)\rangle.\end{aligned} \tag{3.0.4}$$

In fact the construction of the $Q$ operator repeates the similar construction of Pasquier and Gaudin [2]. It should be noted that the building of $Q$ operator follows the main line suggested by Baxter [1].

The operator $t(\lambda) \equiv \mathrm{Tr}\,T(\lambda)$, where

$$T(\lambda) \equiv L_1(\lambda + c_1)\cdots L_n(\lambda + c_n)$$

$$L_k(\alpha_k) = \eta\cdot\begin{pmatrix}\alpha_k + x_k\partial_k + s_k & -\partial_k\\ x_k^2\partial_k + 2s_kx_k & \alpha_k - x_k\partial_k - s_k\end{pmatrix} \qquad \alpha_k = \frac{\lambda + c_k}{\eta}$$

is invariant with respect to transformation of the local matrices $L_k$ [1]:

$$L_k \quad\to\quad \bar{L}_k \equiv N_k^{-1}L_kN_{k+1} \qquad N_{n+1} \equiv N_1$$

where the $N_k$ are matrices with scalar elements. Simple calculation shows that the matrix elements of the transformed matrix

$$\bar{L}_k \equiv N_k^{-1}L_kN_{k+1} = \eta\cdot\begin{pmatrix}\bar{L}_k^{11} & \bar{L}_k^{12}\\ \bar{L}_k^{12} & \bar{L}_k^{12}\end{pmatrix} \qquad N_k = \begin{pmatrix}0 & 1\\ -1 & y_k\end{pmatrix}$$

have the form

$$\bar{L}_k^{11} = -(x_k - y_k)^{1+\alpha_k-s_k}\partial_k(x_k - y_k)^{s_k-\alpha_k}$$
$$\bar{L}_k^{12} = -(x_k - y_k)^{1+\alpha_k-s_k}(x_k - y_{k+1})^{1-\alpha_k-s_k}\partial_k(x_k - y_k)^{s_k-\alpha_k}(x_k - y_{k+1})^{s_k+\alpha_k}$$
$$\bar{L}_k^{21} = \partial_k \qquad \bar{L}_k^{22} = (x_k - y_{k+1})^{1-\alpha_k-s_k}\partial_k(x_k - y_{k+1})^{\alpha_k+s_k}.$$

This expression for the $\bar{L}$ operator suggests the function

$$\phi_k(\alpha_k; x_k; y_k, y_{k+1}) \equiv (x_k - y_k)^{\alpha_k-s_k}(x_k - y_{k+1})^{-\alpha_k-s_k}.$$

The operators $\bar{L}_k^{ij}$ act on this function as follows:

$$\bar{L}_k^{11}\phi_k(\alpha_k) = (\alpha_k + s_k)\phi_k(\alpha_k + 1) \qquad \bar{L}_k^{12}\phi_k(\alpha_k) = 0$$
$$\bar{L}_k^{22}\phi_k(\alpha_k) = (\alpha_k - s_k)\phi(\alpha_k - 1).$$

Let us fix the dependence on the $x$-variables in the kernel of the operator $\hat{Q}(\lambda)$ in the form

$$Q(\lambda; x) \leftrightarrow \prod_{k=1}^{n} \phi(\alpha_k; x_k; y_k, y_{k+1}) \qquad \eta\alpha_k = \lambda + c_k$$

where $\{y_i\}$ is now the set of arbitrary parameters. Then we have

$$t(\lambda)Q(\lambda; x; y) = \operatorname{Tr}\prod_{k=1}^{n} \bar{L}_k\phi_k(\alpha_k) = \prod_{k=1}^{n} \begin{pmatrix} \eta(\alpha_k + s_k)\phi_k(\alpha_k + 1) & 0 \\ \ldots & \eta(\alpha_k - s_k)\phi_k(\alpha_k - 1) \end{pmatrix}.$$

After multiplication of these triangular matrices and calculation of the trace, we obtain the 'right' Baxter's relation:

$$t(\lambda)Q(\lambda; x) = \Delta_+(\lambda)Q(\lambda + \eta; x) + \Delta_-(\lambda)Q(\lambda - \eta; x).$$

In the next step we fix the dependence on the $z$-variables in the kernel of the operator $\hat{Q}(\lambda)$ to obtain the 'left' Baxter's relation:

$$Q(\lambda; x, z)t(\lambda) = \Delta_+(\lambda)Q(\lambda + \eta; x, z) + \Delta_-(\lambda)Q(\lambda - \eta; x, z).$$

The rules (3.0.4) allow us to move the $SL(2)$-generators from the function $\Psi(z)$ to the kernel of the $Q$ operator:

$$\langle RQ(\lambda; x, z)|L_1 \ldots L_n\Psi(z)\rangle = \langle RL'_1 \ldots L'_n Q(\lambda; x, z)|\Psi(z)\rangle$$

where

$$L'_k \equiv \eta \cdot \begin{pmatrix} \alpha_k - S_k & -S_k^- \\ -S_k^+ & \alpha_k + S_k \end{pmatrix} = \left[\sigma_2 \cdot L_k\sigma_2\right]^{\mathrm{t}}$$

and 't' means transposition. Then we transform the trace of the product of the $L'$ matrices

$$\operatorname{Tr} L'_1 \cdots L'_n = \operatorname{Tr}\left[L_n \cdots L_1\right]^{\mathrm{t}} = \operatorname{Tr} L_n \cdots L_1$$

and finally obtain

$$\langle RQ(\lambda; x, z)|\operatorname{Tr}\left[L_1 \cdots L_n\right]\Psi(z)\rangle = \langle R\operatorname{Tr}\left[L_n \cdots L_1\right]Q(\lambda; x, z)|\Psi(z)\rangle. \tag{3.0.5}$$

Now it is possible to repeat all calculations as for the the $x$-variable case. Only one obvious modification, related to the opposite ordering of $L$ matrices on the right-hand side of (3.0.5), is required: the new auxiliary parameters $v_k$ have to be ordered in the same 'opposite way'. The final result is as follows:

$$Q(\lambda; z) \leftrightarrow \prod_{k=1}^{n} \phi(\alpha_k; z_k; v_k, v_{k-1}).$$

So far we have concentrated on the first property of $Q$ operator, namely Baxter's equation, and obtain the result that the operator with general kernel of the type

$$Q(\lambda; x, z) = \prod_{k=1}^{n} \int \mathrm{d}y_k\, \mathrm{d}v_k\, \phi(\alpha_k; x_k; y_k, y_{k+1})\Gamma(y, v)\phi(\alpha_k; z_k; v_k, v_{k-1})$$

obeys this equation. In the next step the function $\Gamma(y, v)$ is determined from the commutativity requirement:

$$\hat{Q}(\mu)\hat{Q}(\lambda) = \hat{Q}(\lambda)\hat{Q}(\mu) \qquad t(\mu)\hat{Q}(\lambda) = \hat{Q}(\lambda)t(\mu).$$

In what follows we shall concentrate on the case of the homogeneous $XXX$ chain, where the function $\Gamma(y, v)$ has the simplest form.

*3.1. The $Q$ operator for the homogeneous $XXX$ chain*

In this section we consider the homogeneous $XXX$ chain of equal spins: $c_i = 0,\ s_i = s$. The kernel

$$Q(\lambda; x, z) \equiv (-1)^{-2sn} \prod_{k=1}^{n} (x_k - z_k)^{-(\eta s-\lambda)/\eta} (x_k - z_{k+1})^{-(\eta s+\lambda)/\eta}$$

has the 'true' $x$- and $z$-dependences and therefore the $Q$ operator can be defined as follows:

$$\hat{Q}(\lambda)\Psi(x) = \langle RQ(\lambda; x, z)|\Psi(z)\rangle = \prod_{k=1}^{n} \langle (1 - z_k x_k)^{-(\eta s-\lambda)/\eta} (1 - z_k x_{k-1})^{-(\eta s+\lambda)/\eta} |\Psi(z)\rangle.$$

There are some useful integral representations for the $Q$ operator obtained here.

*3.2. The $\alpha$ representation for the $Q$ operator*

Let us consider the $Q$ operator

$$\hat{Q}(\lambda)\Psi(x_1, \ldots, x_n) \equiv \prod_{k=1}^{n} \langle (1 - x_{k-1} z_k)^{-(\eta s+\lambda)/\eta} (1 - x_k z_k)^{-(\eta s-\lambda)/\eta} |\Psi(z_1, \ldots, z_n)\rangle$$

and transform the $z_k$ integral using the following identity:

$$\begin{aligned} &\int_{|z_k|\leqslant 1} Dz_k\ (1 - x_k \bar{z}_k)^{-a} (1 - x_{k-1}\bar{z}_k)^{-b} \Psi(z_k) \\ &\qquad = \frac{\Gamma(2s)}{\Gamma(a)\Gamma(b)} \int_0^1 \mathrm{d}\alpha\, \alpha^{a-1} (1-\alpha)^{b-1} \Psi\big[\alpha x_k + (1-\alpha)x_{k-1}\big] \\ &\qquad\qquad a + b = 2s. \end{aligned} \tag{3.2.1}$$

To prove this identity we use the Feynman formula

$$\frac{1}{A^a B^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 \mathrm{d}\alpha\, \alpha^{a-1}(1-\alpha)^{b-1} \frac{1}{\big[\alpha A + (1-\alpha)B\big]^{a+b}} \tag{3.2.2}$$

and transform the product

$$(1 - x_k\bar{z}_k)^{-a}(1 - x_{k-1}\bar{z}_k)^{-b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 \mathrm{d}\alpha\, \frac{\alpha^{a-1}(1-\alpha)^{b-1}}{\big[1 - (\alpha x_k + (1-\alpha)x_{k-1})\bar{z}_k\big]^{2s}}.$$

The remaining $z$ integral can easily be calculated:

$$\int_{|z_k|\leqslant 1} Dz_k\ (1 - x\bar{z}_k)^{-2s}\Psi(z_k) = \Psi(x) \qquad x \equiv \alpha x_k + (1-\alpha)x_{k-1}.$$

Finally we obtain the useful integral representation ($\alpha$ representation) for the $Q$ operator:

$$\hat{Q}(\lambda)\Psi(x) \equiv \prod_{k=1}^{n} \Gamma(\lambda; s) \int_0^1 \mathrm{d}\alpha_k\, \alpha_k^{(\eta s-\lambda)/\eta - 1} \bar{\alpha}_k^{(\eta s+\lambda)/\eta - 1} \Psi\big[\ldots, \alpha_k x_k + \bar{\alpha}_k x_{k-1}, \ldots\big] \tag{3.2.3}$$

where $\bar{\alpha} \equiv 1 - \alpha$ and

$$\Gamma(\lambda; s) \equiv \frac{\Gamma(2s)}{\Gamma(s + \lambda\eta^{-1})\Gamma(s - \lambda\eta^{-1})}.$$

Let us consider the eigenvalue problem for the $Q$ operator

$$\hat{Q}(\lambda)\Psi(x) = Q(\lambda)\Psi(x)$$

where the polynomial $\Psi(x)$ belongs to the space of homogeneous polynomials of degree $l$ (2.2.8):

$$\Psi(x) = \sum_p \Psi_{p_1,\dots,p_n} x_1^{p_1} \dots x_n^{p_n} \qquad p_1 + p_2 + \cdots + p_n = l \quad l = 0, 1, 2, \dots . \tag{3.2.4}$$

The $Q$ operator transforms polynomial $\Psi(x)$ to homogeneous polynomial of degree $l$ whose coefficients are polynomials in $\lambda$ of degree $l$. Therefore eigenvalues $Q(\lambda)$ of the $Q$ operator are polynomials in $\lambda$ of degree $l$.

For the proof we use obtained $\alpha$ representation. Let us consider the action of the $Q$ operator on the polynomial $\Psi(x)$:

$$\hat{Q}(\lambda)\Psi(x) \equiv \sum_p \Psi_{p_1,\dots,p_n} \prod_{k=1}^n \Gamma(\lambda; s) \int_0^1 \mathrm{d}\alpha_k \, \alpha_k^{(\eta s-\lambda)/\eta-1} \bar{\alpha}_k^{(\eta s+\lambda)/\eta-1} \left[\alpha_k x_k + \bar{\alpha}_k x_{k-1}\right]^{p_k}.$$

The expression for the $\alpha_k$ integral have the form

$$\Gamma(\lambda; s) \int_0^1 \mathrm{d}\alpha_k \, \alpha_k^{(\eta s-\lambda)/\eta-1} \bar{\alpha}_k^{(\eta s+\lambda)/\eta-1} \left[\alpha_k x_k + \bar{\alpha}_k x_{k-1}\right]^{p_k} = \sum_{m=0}^{p_k} C_{p_k,m} x_k^m x_{k-1}^{p_k-m}$$

where the coefficients

$$C_{p_k,m} = \frac{p_k!}{m!(p_k-m)!} \frac{\Gamma(2s)}{\Gamma(2s+p_k)} \frac{\Gamma(s-\lambda\eta^{-1}+m)}{\Gamma(s-\lambda\eta^{-1})} \frac{\Gamma(s+\lambda\eta^{-1}+p_k-m)}{\Gamma(s+\lambda\eta^{-1})}$$

are polynomials in $\lambda$ of degree $p_k$ because of evident equality

$$\frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)\cdots(a+m-1).$$

There are similar expressions for the remaining $\alpha$ integrals and we obtain the result that the $Q$ operator transforms the polynomial $\Psi(x)$ to the homogeneous polynomial of degree $l$ whose coefficients are polynomials in $\lambda$ of degree $p_1 + p_2 + \cdots p_n = l$.

There exists some another useful representation for the $Q$ operator ($t$ representation):

$$\hat{Q}(\lambda)\Psi(x) \equiv \prod_{k=1}^n \frac{\Gamma(\lambda; s)}{(x_k - x_{k-1})^{2s-1}} \times \int_{x_{k-1}}^{x_k} \mathrm{d}t_k \, (t_k - x_{k-1})^{(\eta s-\lambda)/\eta-1} (x_k - t_k)^{(\eta s+\lambda)/\eta-1} \Psi[\dots t_k \dots]. \tag{3.2.5}$$

This formula is obtained from (3.2.3) by the following change of variables:

$$t_k = \alpha_k x_k + \bar{\alpha}_k x_{k-1}.$$

### *3.3. SL(2)-invariance and commutativity of the Q operator*

We shall prove two important properties of the $Q$ operator obtained above: $SL(2)$-invariance and commutativity. Let us begin from $SL(2)$-invariance

$$S\hat{Q}(\lambda)\Psi(x) = \hat{Q}(\lambda)S\Psi(x) \qquad S\Psi(x) \equiv (cx+d)^{-2s}\Psi(Sx) \qquad Sx \equiv \frac{ax+b}{cx+d}.$$

The simplest way is to use the representation (3.2.5). We start from $S\hat{Q}(\lambda)$:

$$S\hat{Q}(\lambda)\Psi(x) \sim (Sx_k - Sx_{k-1})^{-2s+1} \times \int_{Sx_{k-1}}^{Sx_k} \mathrm{d}t \, (t - Sx_{k-1})^{(\eta s-\lambda)/\eta-1} (Sx_k - t)^{(\eta s+\lambda)/\eta-1} \Psi(t)$$

and make the following changes of variable in the $t$ integral:

$$t = S\tau = \frac{a\tau + b}{c\tau + d} \qquad S\tau - Sx = \frac{\tau - x}{(c\tau + d)(cx + d)} \qquad \mathrm{d}t = \frac{\mathrm{d}\tau}{(c\tau + d)^2}.$$

After these changes of variable the $t$ integral is transformed to the $\tau$ integral of the required form:

$$(x_k - x_{k-1})^{-2s+1} \int_{x_{k-1}}^{x_k} \mathrm{d}\tau \, (\tau - x_{k-1})^{(\eta s-\lambda)/\eta - 1} (x_k - \tau)^{(\eta s+\lambda)/\eta - 1} (c\tau + d)^{-2s} \Psi(S\tau)$$
$$\sim \hat{Q}(\lambda) S\Psi(x).$$

It is worth emphasizing that all the factors like $(cx_k + d)^{-2s}$ are cancelled in the whole product.

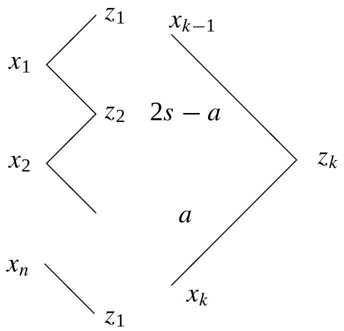
The second important property of the $Q$ operator is commutativity:

$$\hat{Q}(\mu)\hat{Q}(\lambda) = \hat{Q}(\lambda)\hat{Q}(\mu). \tag{3.3.1}$$

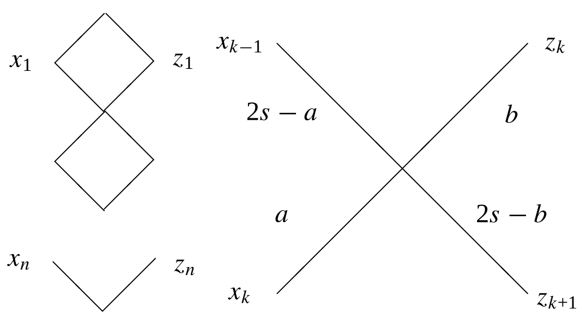
It follows that there exists a unitary operator $U$ independent of $\lambda$ which diagonalizes $\hat{Q}(\lambda)$ simultaneously for all values of $\lambda$, and therefore due to Baxter's relation the operators $\hat{Q}(\lambda)$ and $t(\mu)$ also commute:

$$t(\mu)\hat{Q}(\lambda) = \hat{Q}(\lambda)t(\mu). \tag{3.3.2}$$

It is useful to visualize the $Q$ operator itself and the product of the two $Q$ operators as shown in the following diagram: the line with index $a$ between the points $x$ and $z$ represents the function $(1 - xz)^{-a}$ where $a = (\eta s - \lambda)/\eta$ and $b = (\eta s - \mu)/\eta$. The integration (3.0.3) in any four-point vertex is assumed.




$Q(\lambda; x; z)$

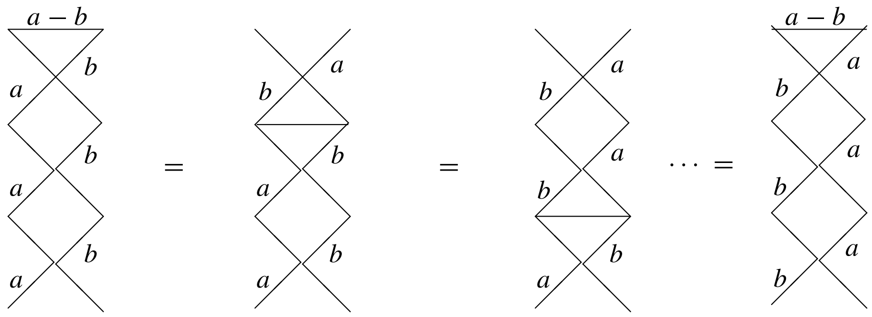



$\hat{Q}(\lambda)\hat{Q}(\mu)$

Let us consider the product $\hat{Q}(\lambda)\hat{Q}(\mu)$ and the corresponding kernel

$$\langle Q(\lambda; x; y) | Q(\mu; y; z)\rangle$$
$$\equiv \prod_{k=1}^{n} \left\langle (1 - x_{k-1}y_k)^{(-\lambda-\eta s)/\eta} (1 - x_k y_k)^{(\lambda-\eta s)/\eta} \middle| (1 - y_k z_{k+1})^{(\mu-\eta s)/\eta} (1 - y_k z_k)^{(-\mu-\eta s)/\eta} \right\rangle.$$

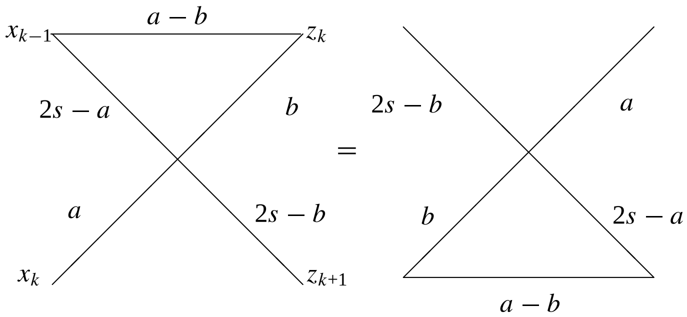
The 'mechanism' of commutativity is shown below [2]:




and is grounded on the 'local' identity

$$\langle (1-x_{k-1}y)^{-2s+a}(1-x_k y)^{-a}|(1-yz_k)^{-b}(1-yz_{k+1})^{-2s+b}\rangle \cdot (1-z_k x_{k-1})^{a-b}$$
$$= \left\langle (1-x_{k-1}y)^{-2s+b}(1-x_k y)^{-b}\middle|(1-yz_k)^{-a}(1-yz_{k+1})^{-2s+a}\right\rangle$$
$$\cdot (1-z_{k+1}x_k)^{a-b}. \tag{3.3.3}$$

The graphic representation of this identity ($a$, $b$ are arbitrary parameters) is




The proof of the equality (3.3.3) can be found in the appendix.

### *3.4. Eigenvalues of the Q operator for n = 2*

The case $n = 2$ is the simplest one:

$$t(\lambda) = 2\lambda^2 + 2\eta^2\vec{S}_1\vec{S}_2 = 2\lambda^2 - 2\eta^2 s(s-1) + \eta^2 L.$$

There exists only one integral of motion, namely the two-particle Casimir $L \equiv (\vec{S}_1 + \vec{S}_2)^2$. Its highest-weight eigenfunctions have the form

$$\Psi_l(x_1, x_2) = (x_1 - x_2)^l \qquad L\Psi_l = (l+2s)(l+2s-1)\Psi_l.$$

Due to $SL(2)$-invariance these functions are eigenfunctions for the $Q$ operator also. Let us calculate the eigenvalue $Q_l(\lambda)$:

$$\hat{Q}(\lambda)\Psi_l = Q_l(\lambda)\cdot\Psi_l.$$

The simplest way is to use the $\alpha$ representation

$$\hat{Q}(\lambda)\Psi_l \equiv \Gamma^2(\lambda; s)\int_0^1 \mathrm{d}\alpha\,\mathrm{d}\beta\,(\alpha\beta)^{(\eta s-\lambda)/\eta-1}(\bar{\alpha}\bar{\beta})^{(s+\lambda)/\eta-1}\cdot\Psi_l\left[\alpha x_1 + \bar{\alpha}x_2; \beta x_2 + \bar{\beta}x_1\right]$$

so that we obtain

$$Q_l(\lambda) = (-1)^l\Gamma^2(\lambda; s)\int_0^1 \mathrm{d}\alpha\,\mathrm{d}\beta\,(\alpha\beta)^{(\eta s-\lambda)/\eta-1}(\bar{\alpha}\bar{\beta})^{(s+\lambda)/\eta-1}(1-\alpha-\beta)^l.$$

The eigenvalue $Q_l(\lambda)$ was obtained in equivalent form in the paper [7] and the polynomials (in $\lambda$) $Q_l(\lambda)$ coincide with the Hahn orthogonal polynomials.

## 4. The $Q$ operator for $\lambda = \pm\eta s$ and local Hamiltonians

Let us consider the $Q$ operator in the $\alpha$ representation:

$$\hat{Q}(\lambda)\Psi(x) \equiv \prod_{k=1}^{n} \Gamma(\lambda; s) \int_0^1 \mathrm{d}\alpha_k\, \alpha_k^{(\eta s-\lambda)/\eta-1} \bar{\alpha}_k^{(\eta s+\lambda)/\eta-1} \Psi\big[\ldots \alpha_k x_k + \bar{\alpha}_k x_{k-1} \ldots\big]$$

for the special value of the spectral parameter $\lambda = \eta s + \eta\epsilon$, and calculate the first two terms of the $\epsilon$ expansion.

We start from the $\alpha_k$ integral

$$\frac{\Gamma(2s)}{\Gamma(2s+\epsilon)\Gamma(-\epsilon)} \int_0^1 \mathrm{d}\alpha\, \alpha^{-\epsilon-1} \bar{\alpha}^{2s+\epsilon-1} \Psi\big[\ldots \alpha x_k + \bar{\alpha} x_{k-1} \ldots\big].$$

The prefactor in this expression is proportional to $\epsilon$ and there is a singular $\epsilon$-pole term in the $\alpha$ integral because of the singularity at the point $\alpha = 0$. For the calculation of the $\epsilon$-pole term one can put $\alpha = 0$ in the argument of the $\Psi$ function:

$$\int_0^1 \mathrm{d}\alpha\, \alpha^{-\epsilon-1} \bar{\alpha}^{2s+\epsilon-1} \Psi\big[\alpha x_k + \bar{\alpha} x_{k-1}\big] \to \frac{\Gamma(-\epsilon)\Gamma(2s+\epsilon)}{\Gamma(2s)} \cdot \Psi\big[x_{k-1}\big].$$

In the main order of the $\epsilon$ expansion we need the singular part of the integral alone, and have

$$\hat{Q}(\eta s)\Psi(x_1, x_2, \ldots, x_k, \ldots, x_n) = \Psi(x_n, x_1, \ldots, x_{k-1}, \ldots, x_{n-1}). \qquad (4.0.1)$$

Therefore the $Q$ operator for $\lambda = \eta s$ coincides with the 'shift' operator $P$:

$$P\Psi(x_1, x_2, \ldots, x_k, \ldots, x_n) = \Psi(x_n, x_1, \ldots, x_{k-1}, \ldots, x_{n-1}) \qquad \hat{Q}(\eta s) = P.$$

In the next order of the $\epsilon$ expansion we have to extract the $\epsilon$-pole contributions from the $n-1$ $\alpha$ integrals and the next term of the $\epsilon$ expansion from the one remaining integral. This remaining $\alpha_k$ integral has the form

$$\frac{\Gamma(2s)}{\Gamma(2s+\epsilon)\Gamma(-\epsilon)} \int_0^1 \mathrm{d}\alpha_k\, \alpha_k^{-\epsilon-1} \bar{\alpha}_k^{2s+\epsilon-1} \Psi(\ldots x_{k-2}, \alpha_k x_k + \bar{\alpha}_k x_{k-1}, x_k \ldots).$$

Note that $\epsilon$-pole contributions effectively shift all arguments of the $\Psi(x_1, \ldots, x_n)$ function except for the $k$th one. In calculating the $\alpha_k$ integral it is useful to add and subtract the pole term:

$$\frac{\Gamma(2s)}{\Gamma(2s+\epsilon)\Gamma(-\epsilon)} \int_0^1 \mathrm{d}\alpha\, \alpha^{-\epsilon-1} \bar{\alpha}^{2s+\epsilon-1} \left[\Psi(\alpha x_k + \bar{\alpha} x_{k-1}) \pm \Psi(x_{k-1})\right].$$

The integral with the difference is regular so we can put $\epsilon = 0$ in the integrand and extract the required contribution:

$$-\epsilon \int_0^1 \mathrm{d}\alpha\, \frac{\bar{\alpha}^{2s-1}}{\alpha} \left[\Psi(\alpha x_k + \bar{\alpha} x_{k-1}) - \Psi(x_{k-1})\right] + \Psi(x_{k-1}).$$

Finally we obtain the first two terms in the $\epsilon$ expansion of the $Q$ operator:

$$\hat{Q}(\eta s + \eta\epsilon) = P + \epsilon \sum_{k=1}^{n} H^{-}_{k-1,k} + \mathrm{O}(\epsilon^2)$$

where the operator $H^{-}_{k-1,k}$ is defined as follows:

$$\begin{aligned} &H^{-}_{k-1,k}\Psi(x_1, \ldots, x_k, \ldots, x_n) \\ &\quad = -\int_0^1 \mathrm{d}\alpha\, \frac{\bar{\alpha}^{2s-1}}{\alpha} \big[\Psi(\ldots x_{k-2}, \alpha x_k + \bar{\alpha} x_{k-1}, x_k \ldots) \\ &\qquad - \Psi(\ldots x_{k-2}, x_{k-1}, x_k \ldots)\big]. \end{aligned}$$

Note this 'two-particle' operator is not $SL(2)$-invariant.

In a similar way one can calculate the first two terms of the $\epsilon$ expansion for $\lambda = -\eta s$:

$$\hat{Q}(-\eta s + \eta\epsilon) = 1 - \epsilon \sum_{k=1}^{n} H^{+}_{k-1,k} + \mathrm{O}(\epsilon^2)$$

where

$$\begin{aligned} H^{+}_{k-1,k}\Psi(x_1, \ldots, x_k, \ldots, x_n) &= -\int_0^1 \mathrm{d}\alpha \, \frac{\bar{\alpha}^{2s-1}}{\alpha} \big[\Psi(\ldots x_{k-1}, \bar{\alpha}x_k + \alpha x_{k-1}, x_{k+1} \ldots) \\ &\quad - \Psi(\ldots x_{k-1}, x_k, x_{k+1} \ldots)\big]. \end{aligned}$$

Let us consider the $\epsilon$ expansion of the following combination of $Q$ operators:

$$\hat{Q}^{-1}(\eta s)\hat{Q}(\eta s + \eta\epsilon) - \hat{Q}^{-1}(-\eta s)\hat{Q}(-\eta s + \eta\epsilon) = \epsilon \sum_{k=1}^{n} H_{k-1,k} + \mathrm{O}(\epsilon^2).$$

Using the expressions for the operators $H^{-}_{k-1,k}$ and $H^{+}_{k-1,k}$ it is easy to check that the operator $H_{k-1,k}$ acts only on the variables $z_{k-1}, z_k$ and coincides with the integral operator considered in (2.3.3). Finally, we have found the following operator relations:

- $\hat{Q}(-\eta s) = 1$
- $\hat{Q}(\eta s) = P$
- $\hat{Q}^{-1}(\eta s)\hat{Q}(\eta s + \eta\epsilon) - \hat{Q}^{-1}(-\eta s)\hat{Q}(-\eta s + \eta\epsilon) = \epsilon H + \mathrm{O}(\epsilon^2); \quad H \equiv \sum_{k=1}^{n} \mathrm{H}_{k-1,k}$.

Let us compare these relations with those obtained by the ABA method, namely equations (2.3.4), (2.3.6). The first relation fixes the normalization of the $Q$ operator and the normalization of the eigenvalues of the $Q$ operator:

$$Q(\lambda) = \prod_{j=1}^{l} \frac{\lambda - \lambda_j}{-\eta s - \lambda_j}. \tag{4.0.2}$$

The second relation allows one to express the eigenvalues of the 'shift' $P$ operator in terms of the function $Q(\lambda)$:

$$P_l = Q(\eta s) = \prod_{j=1}^{l} \frac{\lambda_j - \eta s}{\lambda_j + \eta s}$$

in agreement with (2.3.6). The third relation is the operator version of the equality (2.3.4).

## 5. Asymptotic expansion of the $Q$ operator for $\lambda \to \infty$

Lipatov [9] has found some beautiful symmetry in the $XXX$ model, namely the duality transformation. In this section we show that the duality operator $\mathcal{S}$ arises naturally as the leading term in the asymptotic of the $Q$ operator for large $\lambda$.

To start with let us define a transformation which is analogous to the Fourier tranformation from the coordinate representation to the momentum representation.

### 5.1. The momentum representation

Let us define the transformation $T$ from the function $\bar{\Psi}(x)$ in the 'momentum' representation to the function $\Psi(x)$ in the 'coordinate' representation used up to this point:

$$\Psi(x) = T\left[\bar{\Psi}(x)\right] \qquad \Psi(x_1,\ldots,x_n) \equiv \bar{\Psi}(\partial_{a_1},\ldots,\partial_{a_n}) \prod_{k=1}^{n} \frac{1}{[1-a_k x_k]^{2s}}\bigg|_{a=0}.$$

This transformation maps polynomials to polynomials and can be represented as a combination of Laplace transformation and inversion:

$$T\left[\bar{\Psi}(x)\right] = \frac{1}{\Gamma(2s)} R \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-tx} t^{2s-1} \bar{\Psi}(t).$$

Using the well known properties of the Laplace transformation and (3.0.4), it is easy to derive the expression for the $SL(2)$-generators in the 'momentum' representation:

$$T\left[x\bar{\Psi}(x)\right] = [x^2\partial + 2sx]\Psi(x)$$

$$T\left[(x\partial^2 + 2s\partial)\bar{\Psi}(x)\right] = \partial\Psi(x)$$

$$T\left[(x\partial + s)\bar{\Psi}(x)\right] = (x\partial + s)\Psi(x).$$

To obtain the rules for the transformation of the commuting operators $Q_k$ (2.2.3) from one representation to the other, we start from the very beginning and consider the transformation of the $L$ operator. The $L$ operator in the coordinate representation is the $T$-transformation from the $L'$ operator in the 'momentum' representation:

$$L = \begin{pmatrix} \lambda + \eta[x\partial + s] & -\eta\partial \\ \eta[x^2\partial + 2s\partial] & \lambda - \eta[x\partial + s] \end{pmatrix}$$

$$L' = \begin{pmatrix} \lambda + \eta[x\partial + s] & -\eta[x\partial^2 + 2s\partial] \\ \eta x & \lambda - \eta[x\partial + s] \end{pmatrix} = \sigma_2 \bar{L} \sigma_2$$

where

$$\bar{L} \equiv \begin{pmatrix} \lambda - \eta[x\partial + s] & -\eta x \\ \eta[x\partial^2 + 2s\partial] & \lambda + \eta[x\partial + s] \end{pmatrix}.$$

The $\sigma_2$ matrices are cancelled for the transfer matrix, and we can work directly with $\bar{L}$.

Finally we have the formal rules for the transformation from one representation to the other:

$$\eta \to -\eta \qquad x \to \partial$$

and the operators $x$ and $\partial$ have to be 'normal ordered': all $\partial$'s stay on the right of the $x$'s.

### 5.2. Asymptotic expansion for large $\lambda$

Let us calculate the asymptotic of the $Q$ operator

$$\hat{Q}(\lambda)\Psi(x) = \prod_{k=1}^{n} \Gamma(\lambda; s) \int_0^1 \mathrm{d}\alpha_k\, \alpha_k^{(\eta s-\lambda)/\eta - 1} \bar{\alpha}_k^{(\eta s+\lambda)/\eta - 1} \Psi\left[\ldots \alpha_k x_k + \bar{\alpha}_k x_{k-1} \ldots\right]$$

for large values of the spectral parameter $\lambda$. Without loss of generality we can restrict the full space of polynomials to the subspace of homogeneous (degree $l$) polynomials $\Psi(x)$ (3.2.4).

There is an expression for the $Q$ operator which is more useful in calculating the asymptotic:

$$\hat{Q}(\lambda)\Psi(x) = \bar{\Psi}(\partial_a) \prod_{k=1}^{n} \frac{1}{[1-a_k x_k]^{s-\lambda\eta^{-1}}[1-a_k x_{k-1}]^{s+\lambda\eta^{-1}}}\bigg|_{a=0}. \qquad (5.2.1)$$

This formula can be obtained as follows:

$$\int_0^1 \mathrm{d}\alpha\, \alpha^{(\eta s-\lambda)/\eta-1}\bar{\alpha}^{(\eta s+\lambda)/\eta-1}\Psi\big[\alpha x_k + \bar{\alpha}x_{k-1}\big]$$

$$= \bar{\Psi}[\partial_{a_k}]\int_0^1 \mathrm{d}\alpha\, \frac{\alpha^{(\eta s-\lambda)/\eta-1}\bar{\alpha}^{(\eta s+\lambda)/\eta-1}}{[1-a_k(\alpha x_k+\bar{\alpha}x_{k-1})]^{2s}}$$

$$= \Gamma^{-1}(\lambda; s)\bar{\Psi}[\partial_{a_k}]\frac{1}{[1-a_k x_k]^{s-\lambda\eta^{-1}}\big[1-a_k x_{k-1}\big]^{s+\lambda\eta^{-1}}}$$

where the Feynman formula (3.2.2) is used in the 'opposite' direction.

For the calculation of the asymptotic it is useful to rescale the variables $a_i$ and use the standard expansion for the logarithm:

$$a_i \to \frac{\eta a_i}{\lambda} \qquad \left(1-\frac{ax}{\lambda}\right)^{\lambda-s} = \exp\left\{-ax + \frac{2sax - a^2x^2}{2\lambda} + \cdots\right\}.$$

Let us consider the contribution with $a_k$:

$$\bar{\Psi}(\partial_{a_k})\frac{1}{[1-a_k x_k]^{s-\lambda\eta^{-1}}[1-a_k x_{k-1}]^{s+\lambda\eta^{-1}}}\Bigg|_{a_k=0}$$

$$= \bar{\Psi}(\lambda\eta^{-1}\partial_a)\exp\Bigg\{a(x_{k-1}-x_k)$$

$$+ \frac{\eta(x_{k-1}+x_k)}{2\lambda}\big(2sa + (x_{k-1}-x_k)a^2\big) + \cdots\Bigg\}\Bigg|_{a=0}$$

$$= \bar{\Psi}(\lambda\eta^{-1}z) + \frac{\eta(x_{k-1}+x_k)}{2\lambda}\big(x\partial^2 + 2s\partial\big)\bar{\Psi}(\lambda\eta^{-1}x)\Big|_{x=x_{k-1}-x_k} + \mathrm{O}(\lambda^{-2}).$$

The polynomial $\Psi(x)$ is homogeneous, so that

$$\bar{\Psi}(\lambda\eta^{-1}x) = (\lambda\eta^{-1})^l\bar{\Psi}(x)$$

and finally we obtain the first two terms of the asymptotic expansion of the $Q$ operator for large $\lambda$:

$$\begin{aligned}
&\hat{Q}(\lambda) = \sum_{k=0}^{l}\hat{Q}_k\cdot(\lambda\eta^{-1})^{l-k}\\
&\hat{Q}_0\Psi(x) = \bar{\Psi}(x_n-x_1, x_1-x_2, \ldots, x_{n-1}-x_n)\\
&\hat{Q}_1\Psi(x) = \frac{1}{2}\sum_{k=1}^{n}(x_k+x_{k-1})\big(z_k\partial_{z_k}^2 + 2s\partial_{z_k}\big)\bar{\Psi}(z)\Big|_{z_k=x_{k-1}-x_k}.
\end{aligned} \tag{5.2.2}$$

It seems that all operators $\hat{Q}_k$ are local differential operators in the momentum representation.

The duality transformation operator $\mathcal{S}$ is defined in the following way:

$$\mathcal{S}\Psi(x_1, \ldots, x_n) \equiv \bar{\Psi}(x_n-x_1, x_1-x_2, \ldots, x_{n-1}-x_n).$$

This operator coincides with the leading term $\hat{Q}_0$ of the asymptotic expansion and therefore the operator $\mathcal{S}$ commutes with all integrals of motion $Q_k$ (2.2.3). Its common eigenfunction has to be the eigenfunction of the $\mathcal{S}$ operator:

$$\mathcal{S}\Psi(x_1, \ldots, x_n) = \mathcal{S}_l\cdot\Psi(x_1, \ldots, x_n)$$

$$\Leftrightarrow \quad \bar{\Psi}(x_n-x_1, x_1-x_2, \ldots, x_{n-1}-x_n) = \mathcal{S}_l\cdot\Psi(x_1, \ldots, x_n).$$

Note that the subspace of the common eigenvectors of $Q_k$ with some eigenvalues $q_k$ is the $SL(2)$ module generated by the highest-weight vector $\Psi$. The highest-weight vector $\Psi$ is defined by the equation $S^-\Psi = 0$ and can be constructed by the ABA method (2.2.8). From the expression for the eigenvalue of the $Q$ operator (4.0.2) one can derive the expression for the $\mathcal{S}_l$:

$$Q(\lambda) = \prod_{j=1}^{l} \frac{\lambda - \lambda_j}{-\eta s - \lambda_j} \to \frac{(-\lambda)^l}{\prod_{j=1}^{l}(\eta s + \lambda_j)} \qquad \hat{Q}(\lambda) \to (\lambda\eta^{-1})^l \cdot \mathcal{S}.$$

Therefore the eigenvalue of the duality operator for the highest-weight vector of the $SL(2)$ module has the form

$$\mathcal{S}_l = \frac{(-\eta)^l}{\prod_{j=1}^{l}(\eta s + \lambda_j)}.$$

It is easy to see that all other vectors of the $SL(2)$ module form the 'zero' subspace:

$$\Phi(x_1, \ldots, x_n) = S^+\Psi(x_1, \ldots, x_n)$$
$$\to \quad \mathcal{S}\Phi(x_1, \ldots, x_n) = \mathcal{S}(x_1 + \cdots + x_n)\bar{\Psi}(x_1, \ldots, x_n) = 0.$$

## 6. Conclusions

We have constructed Baxter's $Q$ operator for the homogeneous $XXX$ spin chain and have checked the consistency of the results obtained with the corresponding formulae obtained in the framework of the ABA method. We have found the connection between the $Q$ operator and the duality symmetry operator.

The construction considered here can be applied to the inhomogeneous $XXX$ model, but we are not able to obtain the useful and compact representation for the $Q$ operator in this case.

There exists a more universal approach to the construction of the quantum $Q$ operator. Kuznetsov and Sklyanin informed me that they recently obtained similar results [11] using the approach of [10] based on the correspondence between the quantum $Q$ operator and the classical Bäcklund transformation.

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## Appendix

In this appendix we prove the identity

$$\begin{aligned}\langle(1 - x_{k-1}y)^{-2s+a}(1 - x_k y)^{-a}|(1 - yz_k)^{-b}(1 - yz_{k+1})^{-2s+b}\rangle \cdot (1 - z_k x_{k-1})^{a-b} \\ = \langle(1 - x_{k-1}y)^{-2s+b}(1 - x_k y)^{-b}|(1 - yz_k)^{-a}(1 - yz_{k+1})^{-2s+a}\rangle \\ \times (1 - z_{k+1}x_k)^{a-b}.\end{aligned} \tag{A.1}$$

First of all note that the equality (3.2.1) can be rewritten in integral form:

$$\langle (1-x_{k-1}z)^{-a}(1-x_{k-1}z)^{-b}|\Psi(z)\rangle$$
$$= \frac{\Gamma(2s)}{\Gamma(a)\Gamma(b)}\frac{1}{(x_k-x_{k-1})^{2s-1}}\int_{x_{k-1}}^{x_k}\mathrm{d}t\,(t-x_{k-1})^{b-1}(x_k-t)^{a-1}\Psi(t)$$

and therefore the identity (A.1) is equivalent to the following integral identity:

$$\frac{\Gamma(2s)}{\Gamma(2s-a)\Gamma(a)}\frac{(1-x_{k-1}z_k)^{a-b}}{(x_k-x_{k-1})^{2s-1}}\int_{x_{k-1}}^{x_k}\mathrm{d}t\,\frac{(t-x_{k-1})^{a-1}(x_k-t)^{2s-a-1}}{(1-tz_k)^b(1-tz_{k+1})^{2s-b}}$$
$$= \frac{\Gamma(2s)}{\Gamma(2s-b)\Gamma(b)}\frac{(1-x_kz_{k+1})^{a-b}}{(x_k-x_{k-1})^{2s-1}}$$
$$\times\int_{x_{k-1}}^{x_k}\mathrm{d}\tau\,\frac{(\tau-x_{k-1})^{b-1}(x_k-\tau)^{2s-b-1}}{(1-\tau z_k)^a(1-\tau z_{k+1})^{2s-a}}. \tag{A.2}$$

Let us start from the $t$ integral. There exists the bilinear transformation with the properties

$$z = Sx = \frac{Ax-C}{Cx+D} \qquad Sx_k = z_k \qquad Sx_{k-1} = z_{k+1}.$$

This transformation can be obtained as follows:

$$\frac{z-z_k}{z-z_{k+1}} = R\frac{x-x_k}{x-x_{k-1}} \quad\Rightarrow\quad z = \frac{x(x_k-x_{k-1}R)+z_kx_{k-1}R-x_kz_{k+1}}{x(1-R)+z_kR-z_{k+1}} = \frac{Ax-C}{Cx+D}$$

and therefore

$$A = (x_k-x_{k-1}R) \qquad D = z_kR-z_{k+1} \qquad C = 1-R \qquad R = \frac{1-x_kz_{k+1}}{1-z_kx_{k-1}}.$$

It is worth emphasizing the additional properties

$$Sz_k = x_k\,,\; Sz_{k+1} = x_{k-1} \qquad \frac{Cz_{k+1}+D}{Cz_k+D} = R.$$

Let us make the same bilinear transformation in the $t$ integral:

$$t = S\tau = \frac{A\tau-C}{C\tau+D} \qquad 1-tz_k = \frac{Cz_k+D}{C\tau+D}(1-\tau x_k) \qquad x_k = Sz_k \quad x_{k-1} = Sz_{k+1}.$$

Then we obtain

$$\frac{1}{(x_k-x_{k-1})^{2s-1}}\int_{x_{k-1}}^{x_k}\mathrm{d}t\,\frac{(t-x_{k-1})^{a-1}(x_k-t)^{2s-a-1}}{(1-tz_k)^b(1-tz_{k+1})^{2s-b}}$$
$$= \frac{(Cz_{k+1}+D)^{a-b}}{(Cz_k+D)^{a-b}}\frac{1}{(z_k-z_{k+1})^{2s-1}}\int_{z_{k+1}}^{z_k}\mathrm{d}\tau\,\frac{(\tau-z_{k+1})^{a-1}(z_k-\tau)^{2s-a-1}}{(1-\tau x_k)^b(1-\tau x_{k-1})^{2s-b}}.$$

In the next step we transform the $\tau$ integral using the $\alpha$ representation and the Feynman formula:

$$\frac{1}{(z_k-z_{k+1})^{2s-1}}\int_{z_{k+1}}^{z_k}\mathrm{d}\tau\,\frac{(\tau-z_{k+1})^{a-1}(z_k-\tau)^{2s-a-1}}{(1-\tau x_k)^b(1-\tau x_{k-1})^{2s-b}}$$
$$= \int_0^1\mathrm{d}\alpha\,\frac{\alpha^{a-1}\bar{\alpha}^{2s-a-1}}{[1-(\alpha z_k+\bar{\alpha}z_{k+1})x_k]^b[1-(\alpha z_k+\bar{\alpha}z_{k+1})x_{k-1}]^{2s-b}}$$
$$= \frac{\Gamma(2s)}{\Gamma(b)\Gamma(2s-b)}\int_0^1\mathrm{d}\beta\,\beta^{b-1}\bar{\beta}^{2s-b-1}$$
$$\times\int_0^1\mathrm{d}\alpha\,\frac{\alpha^{a-1}\bar{\alpha}^{2s-a-1}}{[1-(\alpha z_k+\bar{\alpha}z_{k+1})(\beta x_k+\bar{\beta}x_{k-1})]^{2s}}$$

$$= \frac{\Gamma(a)\Gamma(2s-a)}{\Gamma(b)\Gamma(2s-b)} \int_0^1 \mathrm{d}\beta \, \frac{\beta^{b-1}\bar{\beta}^{2s-b-1}}{[1-(\beta x_k + \bar{\beta}x_{k-1})z_k]^a[1-(\beta x_k + \bar{\beta}x_{k-1})z_{k+1}]^{2s-a}}$$

$$= \frac{\Gamma(a)\Gamma(2s-a)}{\Gamma(b)\Gamma(2s-b)} \frac{1}{(x_k - x_{k-1})^{2s-1}} \int_{x_{k-1}}^{x_k} \mathrm{d}\tau \, \frac{(\tau - x_{k-1})^{b-1}(x_k - \tau)^{2s-b-1}}{(1-\tau z_k)^a (1-\tau z_{k+1})^{2s-a}}.$$

Collecting all these together, we obtain the equality (A.2).

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